17/11/22 MATH 4030 Tutorial

Reminders:

- Final: Friday 9 Dec 1530 – 1730 @ University Gymnasium.

Recall:

Def: A regular curve $\alpha \subset M$ is a geodesic if

1) its geodesic curvature $k_g \equiv 0$ $\quad k_g = \langle \alpha'', n \rangle = \langle \alpha'', N \times \alpha' \rangle = \langle \alpha' \times \alpha'', N \rangle$

normal to the curve

unit normal vector field to the surface

2) it is parametrized by arc length.

Prop: α is a geodesic iff in any local coordinates $X(u^1, u^2)$

$$\ddot{u}^k + \sum_{i,j=1}^{2} \Gamma_{ij}^k \dot{u}^i \dot{u}^j = 0 \quad \text{for } k = 1, 2 \qquad (*)$$

where $\dot{u} = \frac{du}{dt}$.

For the surface of revolution $X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$, $f(v)>0$

(*) becomes

$$\begin{cases} u'' + \dfrac{2ff_v}{f^2} u'v' = 0 & (1) \\ v'' - \dfrac{ff_v}{f_v^2+g_v^2}(u')^2 + \dfrac{f_vf_{vv}+g_vg_{vv}}{f_v^2+g_v^2}(v')^2 = 0 & (2) \end{cases}$$

From (1), (2), can see that meridians $(\alpha(t) = X(c,t))$ are geodesics and parallels $(\beta(t) = X(t,c))$ are geodesics if $f_v = 0$.

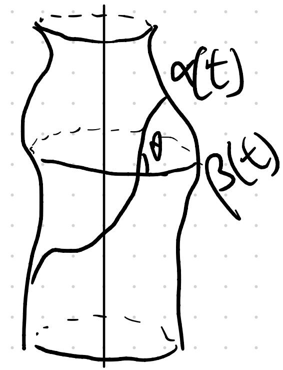
Problem 1: Prove Clairaut's Relation: $\alpha(t)$ is a geodesic intersecting a parallel $\beta(t) = X(t,c)$, c - const. at an angle $\theta(t)$, let r = distance to the axis of revolution. Then

$$r\cos\theta = f(v(t))\cos\theta(t) = \text{const}.$$

Hint: use (1).

$$u'' + \frac{2ff_v}{f^2} u'v' = 0 \iff f^2 u'' + 2ff_v u' = 0 \iff (f^2 u')' = 0.$$

$$\Rightarrow f^2 u' = \text{const}.$$



$$\cos\theta = \frac{\langle \alpha'(t), \beta'(t)\rangle}{\cancel{|\alpha'(t)|}\, |\beta'(t)|} \quad 1$$

$$= \frac{\langle X_u u' + X_v v', X_u\rangle}{|X_u|}$$

$$= \frac{u' \overbrace{|X_u|^2}^{f^2} + v' \overbrace{\cancel{\langle X_u, X_v\rangle}}^{0}}{|X_u| = f}$$

$$= f u'.$$

$\beta(t) = X(t, c)$

$\beta'(t) = X_u$

$\alpha(t) = X(u(t), v(t)).$

$\alpha'(t) = X_u u'(t) + X_v v'(t).$

so $r\cos\theta = f^2 u' = \text{const}$ ✓

p. 257~260 do Carmo.

Pressley (Elementary DG, p183-185).

Converse: if $r\cos\theta = \text{const}$, then either α is a parallel or is a geodesic.

<u>Problem 2</u> do Carmo 4-3 220a For the torus T parametrized by

$$X(u,v) = \big(\underbrace{(r\cos v + a)}_{f}\cos u,\ \underbrace{(r\cos v + a)}_{f}\sin u,\ \underbrace{r\sin v}_{g}\big), \quad 0<r<a$$

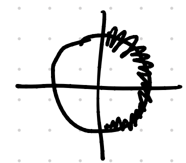
Prove that if a geodesic is tangent to the parallel $v=\frac{\pi}{2}$ at some point (t_0), then it is entirely contained in the region of T given by $-\frac{\pi}{2}\le v\le \frac{\pi}{2}$.

Hint: Use Clairaut's relation.

Pf: Clairaut's relation $\Rightarrow$ $(r\cos v(t) + a)\cos\theta(t) = \text{const.}$

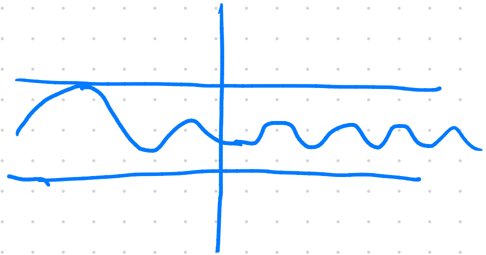
$$\text{at } t_0,\quad LHS = \left(r\cos\left(\tfrac{\pi}{2}\right) + a\right)\cos 0 = a.$$

So $(r\cos v(t) + a)\cos\theta(t) = a$. (1) Now, suppose for contradiction, that there is some t_1 s.t. $v(t_1) \in [-\pi, \pi] \setminus [-\frac{\pi}{2}, \frac{\pi}{2}]$.



$\cos v(t_1) < 0$. (1) rewritten is $\cos\theta(t) = \dfrac{a}{r\cos v(t) + a}$

So then at t_1, $\cos\theta(t_1) = \dfrac{a}{r\cos(v(t_1)) + a} > 1$ a contradiction.



geodesics of the torus.